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On the Parallel Dynamics for the Little–Hopfield Model

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We propose a method (algorithm) for calculation of the explicit formulas for evolution of the main and the residual overlaps. It allows us to confirm the Gardner–Derrida–Mottishaw second-step formula for the main overlap and to go beyond to the next steps. We discuss the dynamical status of the Amit–Gutfreund–Sompolinsky formula for the main overlap and some computer-simulation results.

KEY WORDS: Little–Hopfield model; main and residual overlaps; parallel dynamics.

1. INTRODUCTION

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space of infinite Bernoulli sequences $\{\xi_i\}_{i=1}^{\infty} = \boldsymbol{\xi}$. Here $\Omega = \{-1, 1\}^{\infty}$, $\mathscr{F}$ is the minimal σ-algebra containing all Borel subsets $\mathscr{B}(\mathbb{R}^{\infty}) \cap \Omega$, and $\mathbb{P}$ is a product measure defined on the cylinder sets $C \subset \mathscr{F}$ by

$$\mathbb{P}\{C_{\mathbf{a}_n} = \{\boldsymbol{\xi} \in \mathscr{F} : \xi_{i_1} = a_1, ..., \xi_{i_n} = a_n\}\} = \pi^{\sum_{i=1}^{n}(1+a_i)/2}(1-\pi)^{\sum_{i=1}^{n}(1-a_i)/2} \quad (1.1)$$

where probability $\pi = \mathbb{P}\{\boldsymbol{\xi} \in \mathscr{F} : \xi_j = 1\}$ for arbitrary $j = 1, 2, ...$ Therefore, Ω is the space of realizations of dichotomous independent identically distributed random variables (i.i.d.r.v.).

Let us consider M realizations (trials) of the above Bernoulli sequences of the length N, i.e., $\{\boldsymbol{\xi}_N^p\}_{p=1}^{M} = \{\xi_i^p\}_{i=1, p=1}^{N, M}$ (patterns). Then the Little–Hopfield model[1,2] of the neural network with N neurons and M

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stored uncorrected patterns corresponds to the quenched random Ising model

$$H_N(\mathbf{S}) = -\sum_{\substack{i,j=1\\(i\neq j)}}^{N} J_{ij}^M S_i S_j, \qquad \mathbf{S} = \{S_i\}_{i=1}^N \tag{1.2}$$

where the interaction has the Hebbian form[1,2]

$$J_{ij}^M = \frac{1}{N}\sum_{p=1}^{M} \xi_i^p \xi_j^p \tag{1.3}$$

with $\pi = 1/2$.

Whenever M is not very large, e.g., $M = \alpha \cdot N$ with a small enough α, the ground-state energy $H_n(\mathbf{S})$ has at least $2M$ sound minima separated by barriers of heights $\sim O(N)$. As has been rigorously proven by Newman,[3] there is $\bar{\alpha}_c > 0$ such that for a random system of patterns $\{\xi_N^p\}_{p=1}^M$ the above picture persists for $\alpha \leqslant \bar{\alpha}_c$ in the limit $N \to \infty$ (α-lim) in the sense that the probability of this event converges in this limit to 1.

The aim of the present paper is to add a new result for this model concerning the exact recurrence equations for the parallel dynamics. Here we follow a probabilistic approach to the Little–Hopfield model proposed in ref. 4 and developed in ref. 5. For the multilayered perceptron (Domany–Meir–Kinzel model[6]) this approach gives[7] exact results for the dynamics of the main and the residual overlaps; see also the recent paper by Domany *et al.*[8]

We recall that (for zero temperature $\theta = 0$) the parallel dynamics for the model (1.2), (1.3) corresponds to simultaneous updating the spin configuration $\mathbf{S}$ at the moment $t+1$ according to the rule

$$S_i(t+1) = \operatorname{sign}\left(\sum_{\substack{j=1\\(j\neq i)}}^{N} J_{ij}^M S_j(t)\right), \qquad i = 1, 2,..., N \tag{1.4}$$

The simplest way to take into account nonzero temperature $\theta \neq 0$ is to switch on in (1.4) noisy terms $\{\phi_i\}_{i=1}^{\infty}$ which are i.i.d.r.v. with distributions

$$F_{\phi_i}(x) = \Pr\{\phi_i \leqslant x\} = \frac{1}{2}\left[1 + \operatorname{th}\left(\frac{x}{\theta}\right)\right], \qquad i = 1, 2,... \tag{1.5}$$

so that one gets, instead of (1.4), the following stochastic equations:

$$S_i(t+1) = \operatorname{sign}\left(\sum_{\substack{j=1\\(j\neq i)}}^{N} J_{ij}^M S_j(t) + \phi_i\right), \qquad i = 1, 2,... \tag{1.6}$$

In the next three sections we present formulas describing the first three steps of evolution. Finally, we discuss the status of the Amit–Gudfreund–Sompolinsky (AGS) formula[9] for the main overlap at $t=\infty$. We propose a derivation of this formula based on pure dynamical (instead of thermodynamic[9]) ideas.

2. THE FIRST STEP: KINZEL FORMULA AND THE RESIDUAL OVERLAPS

Let $\{\xi^p\}_{p=1}^M$ be a set of the fixed random patterns (quenched system) realized on the space $(\Omega, \mathscr{F}, \mathbb{P})$. Consider the space $\Omega^M = \prod_{p=1}^M \Omega$ with corresponding σ-algebra $\mathscr{F}^M$ and product measure $\mathbb{P}^M$. Then the above patterns are $\mathbb{P}^M$-almost surely (a.s.) stochastically independent:

$$(\xi^p \cdot \xi^{p'})_N = \frac{1}{N} \sum_{i=1}^N \xi_i^p \xi_i^{p'} \xrightarrow[N\to\infty]{\mathbb{P}^M\text{-a.s.}} 0, \qquad p \neq p' \tag{2.1}$$

Let the initial configuration $\mathbf{S}(t=0)$ be such that

$$m_N^p(t=0) = \frac{1}{N}(\mathbf{S}(0)\cdot\xi^p)_N = \begin{cases} m^q(0) \neq 0, & p=q \\ \xrightarrow[N\to\infty]{\mathbb{P}^m\text{-a.s.}} 0, & p(\neq q) = 1, 2,..., M \end{cases} \tag{2.2}$$

Then, by means of (1.4), *the main overlap* $m^q(\cdot)$ at $t=1$ can be represented as

$$m_N^q(t=1) = \frac{1}{N}\sum_{i=1}^N \operatorname{sign}\left[m_{N,i}^q(t=0) + \xi_i^q \frac{1}{\sqrt{N}} \sum_{\substack{p=1\\(p\neq q)}}^M \xi_i^p r_{N,i}^p(t=0)\right] \tag{2.3}$$

where $\{r^p(\cdot)\}_{p\neq q}$ denotes the *residual* (*noisy*) *overlaps*, i.e.,

$$r_{N,i}^p(t=0) = \frac{1}{\sqrt{N}} \sum_{\substack{j=1\\(j\neq i)}}^N \xi_j^p S_j(t=0) \tag{2.4}$$

The additional subindex i in (2.3), (2.4) means that the corresponding terms in the sums (2.2), (2.4) are canceled,

$$m_{N,i}^q(\cdot) = \frac{1}{N}(\xi^q \cdot \mathbf{S}(\cdot))_{N,i} = \frac{1}{N}\sum_{j\neq i}^N \xi_j^q S_j(\cdot)$$

To establish the dynamics of the main overlap, we have to calculate (2.3) in the α-lim for a *quenched system* of random patterns. Below we show that the result is a.s. (with respect to the measure $\mathbb{P}^\infty$) *independent* of this system.

Therefore, for the first step one has to prove that for an arbitrary fixed set of random patterns the noisy terms $\{\xi_i^q \cdot v_{N,i}^q(t=0)\}_{i=1}^N$, where

$$v_{N,i}^q(t=0) = \frac{1}{\sqrt{N}} \sum_{\substack{p=1 \\ (p \neq q)}}^{M} \xi_i^p r_{N,i}^p(t=0) \tag{2.5}$$

converge to a sequence of i.i.d.r.v. This would allow us to apply to (2.3) the law of large numbers (in the series scheme[10]):

$$\alpha\text{-lim}\, m_N^q(t=1) = E_{\xi v}\, \text{sign}(m^q(t=0) + \eta_i(t=0)) \tag{2.6}$$

where $E_{\xi v}(\cdot)$ is the expectation over the stationary noise $\eta_i(t=0) = \xi_i^q v^q(t=0)$.

To this end, let us remark that according to the initial conditions, one gets that

$$\{\xi_i^p \xi_j^p S_j(t=0)\}_{j=1(j\neq i),\, p=1(p\neq q)}^{N,M} = \|a_{jp}\|$$

is a matrix of i.i.d.r.v. in Ω^M. Hence, by the central limit theorem (CLT) (ref. 10, III.§6) one gets (in the sense of distribution) that

$$\alpha\text{-lim}\, \frac{\lambda_{NM} - E\lambda_{NM}}{(\text{Var}\,\lambda_{NM})^{1/2}} \overset{\text{d}}{=} \mathcal{N}(0, 1) \tag{2.7}$$

Here

$$\lambda_{NM} = \sum_{\substack{j=1 \\ (j\neq i)}}^{N} \sum_{\substack{p=1 \\ (p\neq q)}}^{M} a_{jp}$$

and $\mathcal{N}(0, 1)$ is a Gaussian random variable with mean zero and unit variance. Calculating the expectation and variance in (2.7) and taking into account definition (2.5), we get

$$\alpha\text{-lim}\, v_{N,i}^q(t=0) \overset{\text{d}}{=} \sqrt{\alpha}\, \mathcal{N}(0, 1) \tag{2.8}$$

Then by the symmetry of the distribution $\mathcal{N}(0, 1)$ and the independence of ξ_i^q and $v_{N,i}^q(t=0)$ we obtain the same for $\{\eta_i(t=0)\}_{i=1}^{\infty}$, which are by construction i.i.d.r.v., and $\eta_i(t=0) = \sqrt{\alpha}\, \mathcal{N}(0, 1)$. Consequently [see (2.6)], one gets[11]

$$\begin{aligned} m^q(t=1) &= \text{erf}\left(\frac{m^q(t=0)}{\sqrt{\alpha}}\right) \\ \text{erf}(z) &= \left(\frac{2}{\pi}\right)^{1/2} \int_0^z dx\, \exp\left(\frac{-x^2}{2}\right) \end{aligned} \tag{2.9}$$

To perform the next step $t=2$ one can use the same line of reasoning as above, but now with distribution of $\eta_i(t=1)$. Therefore [see (2.5)], we have to calculate α-lim for residual overlaps $\{r^p(t=1)\}_{p\neq q}$. Using (1.4), one gets the following recurrence relations:

$$r_N^f(t=1)=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\operatorname{sign}\left[\frac{1}{\sqrt{N}}r_{N,i}^f(t=0)+\xi_i^f\left(\xi_i^q m_{N,i}^q(t=0)\right.\right.$$
$$\left.\left.+\frac{1}{\sqrt{N}}\sum_{\substack{p=1\\(p\neq f,q)}}^{M}\xi_i^p r_{N,i}^p(t=0)\right)\right],\qquad f(\neq q)=1,2,\dots,M \qquad (2.10)$$

Let us introduce random variables

$$w_{N,i}^{f,q}(t=0)=\xi_i^q m_{N,i}^q(t=0)+\frac{1}{\sqrt{N}}\sum_{\substack{p=1\\(p\neq f,q)}}^{M}\xi_i^p r_{N,i}^p(t=0) \qquad (2.11)$$

Warning. Random variables $(1/\sqrt{N})r_{N,i}^f(t=0)$ and $\xi_j^f w_{N,j}^{f,q}(t=0)$ are independent for $i=j$, but they are correlated for $i\neq j$, e.g., $(1/\sqrt{N})r_{N,1}^f(t=0)$ is correlated with $\xi_2^f\xi_2^q m_{N,2}^q(t=0)$. So, $\{(1/\sqrt{N})r_{N,i}^f(t=0)+\xi_i^f w_{N,i}^{f,q}(t=0)\}_{i=1}^N$ is the sequence of *dependent* random variables. Compare this with the case of the multilayered perceptron,[7] where they are still independent.

From Eqs. (2.2) and (2.4) by the CLT one gets

$$r^f(t=0)=\alpha\text{-lim } r_{N,i}^f(t=0)=\alpha\text{-lim } r_{N,i}^{f,q}(t=0)\overset{\mathrm{d}}{=}\mathcal{N}(0,1),\qquad f\neq q \qquad (2.12)$$

To derive the recurrence relation for $\{r^p(t=1)\}_{p\neq q}$, one has to fix the realization of residual overlaps at $t=0$ and to consider the noise from choices of the patterns.

Applying the CLT to the i.non-i.d.r.v. $\{\xi_i^p r_{N,i}^p(t=0)\}_{p=1,p\neq f,q}^M$ [according to (2.12), $\{r_{N,i}^p(t=0)\}_p$ is a fixed Gaussian realization; see (2.10)], we obtain

$$\alpha\text{-lim}\frac{1}{\sqrt{N}}\sum_{\substack{p=1\\(p\neq f,q)}}^{M}\xi_i^p r_{N,i}^p(t=0)\overset{\mathrm{d}}{=}\sqrt{\alpha}\,\mathcal{N}(0,1) \qquad (2.13)$$

which is dependent on $\{r^p(t=0)\}$, but independent of $\{\xi^p\}$. Therefore, by (2.11), $\{w_i^q(t=0)\overset{\mathrm{d}}{=}\alpha\text{-lim } w_{N,i}^{f,p}(t=0)\}_{i=1}^{\infty}$ is a sequence of i.i.d.r.v. with probability density

$$p_w^0(x)=\frac{1}{2}\sum_{\sigma=\pm1}\frac{1}{(2\pi\alpha)^{1/2}}\exp\left\{-\frac{[x-\sigma m^q(t=0)]^2}{2\alpha}\right\} \qquad (2.14)$$

By symmetry of (2.14) and independence of ξ_i^q and $w_i^q(t=0)$ we get the same distribution for $\{\xi_i^q w_i^q(t=0)\}_{i=1}^{\infty}$: $p_{\xi w}(x) = p_w^0(x)$.

Now we can to apply the CLT (together with the Berry–Esseen theorem, ref. 10, III. §6, to control the uniformity) to the sequence

$$\left\{\text{sign}\, \delta_{i,N}^f = \text{sign}\left[\frac{1}{\sqrt{N}} r_{N,i}^f(t=0) + \xi_i^f w_{N,i}^{f,q}(t=0)\right]\right\}$$

to get

$$\alpha\text{-lim} \frac{\sum_{i=1}^N \text{sign}\, \delta_{i,N}^f - E\sum_{i=1}^N \text{sign}\, \delta_{i,N}^f}{(\text{Var} \sum_{i=1}^N \text{sign}\, \delta_{i,N}^f)^{1/2}} \stackrel{d}{=} \mathcal{N}(0,1) \tag{2.15}$$

Using (2.14) and $p_{\xi w}(x) = p_w^0(x)$, one gets

$$\alpha\text{-lim} \sqrt{N}\, E\, \text{sign}\, \delta_{i,N}^f = 2p_w^0(x=0)\, r^f(t=0) \tag{2.16}$$

Then we get that

$$\alpha\text{-lim} \frac{1}{N} \text{Var} \sum_{i=1}^N \text{sign}\, \delta_{i,N}^f = 1$$

and finally by Eqs. (2.15) and (2.16) one obtains

$$r^f(t=1) = \mathcal{N}(0,1) + 2p_w^0(x=0)\cdot r^f(t=0), \qquad f \neq q \tag{2.17}$$

Therefore, the limit residual overlaps at the moment $t=1$ are the sum of two *correlated* [see (2.12) and the warning following (2.11)] Gaussian variables.

Using Eqs. (1.4), (2.2) and Eqs. (2.8), (2.9), one can show[5] that α-lim $E(S_k(t=1)\cdot S_k(t=0)) = m^q(t=1)\, m^q(t=0)$. Hence, the covariance is

$$\text{Cov}[r^f(t=1)\cdot r^f(t=0)] = m^q(t=1)\, m^q(t=0) + 2p_w^0(x=0) \tag{2.18}$$

and, as a consequence, we get for the variance of (2.17)

$$D(t=1) \equiv \text{Var}\, r^f(t=1) = 1 + (2p_w^0(x=0))^2 + 4p_w^0(x=0)\, m^q(t=1)\, m^q(t=0) \tag{2.19}$$

So, we are ready to do the second step.

3. THE SECOND STEP: GARDNER–DERRIDA–MOTTISHAW FORMULA AND THE RESIDUAL OVERLAPS

As above, we start with representation [cf. (2.3)]

$$m_N^q(t=2) = \frac{1}{N} \sum_{i=1}^N \text{sign}\left[m_{N,i}^q(t=1) + \xi_i^q \frac{1}{\sqrt{N}} \sum_{\substack{p=1 \\ (p\neq q)}}^M \xi_i^p r_{N,i}^p(t=1)\right] \tag{3.1}$$

To apply the CLT to $\{\xi_i^q v_{N,i}^q(t=1) = \eta_{N,i}^q(t=1)\}_{i=1}^N$ [cf. (2.5) and (2.6)], we first have to consider the convergence of $\{v_{N,i}^q(t=1)\}_{i=1}^N$. To this end, we can apply (for a fixed realization of $S_i(t=0)$) the CLT to i.i.d.r.v. $\{\xi_i^p \cdot r_{N,i}^p(t=1)\}_{p\neq q}$; then

$$\alpha\text{-lim}\ \frac{v_{N,i}^q(t=1) - Ev_{N,i}^q(t=1)}{[\mathrm{Var}\ v_{N,i}^q(t=1)]^{1/2}} \stackrel{\mathrm{d}}{=} \mathcal{N}(0,1) \tag{3.2}$$

On the other hand, according (2.10) and Eqs. (2.11) and (2.14), one gets [see (2.2), (2.17), and (2.19)]

$$\alpha\text{-lim}\ Ev_{N,i}^q(t=1) = S_i(t=0)\, 2\alpha \cdot p_w^0(x=0) \tag{3.3}$$

and $\mathrm{Var}\ v_{N,i}^q(t=1) = \alpha D(t=1) + o(\alpha)$. Therefore, with the help of (3.2) and (3.3), Eq. (3.1) can be represented in the following form:

$$m_N^q(t=2) = \frac{1}{N}\sum_{i=1}^N \mathrm{sign}\{m^q(t=1) + [\alpha D(t=1)]^{1/2}\xi_i^q \mathcal{N}(0,1) + 2\alpha\cdot p_w^0(x=0)\xi_i^q S_i(t=0)\} + o(1) \tag{3.4}$$

Let $\{i = 1, 2,..., N\} = I_+ \cup I_-$, where $I_\pm = \{i\colon \xi_i^q \cdot S_i(t=0) = \pm 1\}$. Then from (3.4) we get, by the ergodic theorem for $I_\pm$ and $\eta_i^q(t=1)$, that in the α-lim

$$m^q(t=2) = \alpha\text{-lim} \sum_{\sigma=\pm 1} \frac{|I_\sigma|}{N}\frac{1}{|I_\sigma|}\sum_{i\in I_\sigma} \mathrm{sign}\{m^q(t=1) + [\alpha D(t=1)]^{1/2}\mathcal{N}(0,1) + 2\alpha\cdot p_w^0(x=0)\sigma\}$$

or finally (Gardner–Derrida–Mottishaw formula[12])

$$m^q(t=2) = \sum_{\sigma=\pm 1} \frac{1+\sigma m^q(t=0)}{2}\,\mathrm{erf}\left[\frac{m^q(t=1) + 2\alpha\sigma p_w^0(x=0)}{[\alpha D(t=1)]^{1/2}}\right] \tag{3.5}$$

Here $p_w^0(x)$ and $D(t=1)$ are defined by (2.14) and (2.19).

Now it is clear that to go beyond this formula to the next step one has first to calculate residual overlaps for $t=2$.

Starting from the explicit formula (2.10) for $t=2$ and the representation of the noisy term $\eta_{N,i}^{f,q}(t=1)$ as we obtain above, one gets

$$r_N^f(t=2) = \frac{1}{\sqrt{N}}\sum_{i=1}^N \mathrm{sign}\left[\frac{1}{\sqrt{N}} r_{N,i}^f(t=1) + \xi_i^f\xi_i^q m^q(t=1) + [\alpha D(t=1)]^{1/2}\xi_i^f\mathcal{N}(0,1) + 2\alpha p_w^0(x=0)\,\xi_i^f S_i(t=0)\right] + o(1) \tag{3.6}$$

Let, for $\sigma_{1,2} = \pm 1$,

$$\Delta(\sigma_1, \sigma_2) = \sigma_1 m^q(t=1) + \sigma_2 2\alpha p_w^0(x=0) \tag{3.7}$$

Then, following the line of reasoning presented above [see (2.11)–(2.14) including the warning following (2.11)], we obtain for the probability density of the noisy terms $\{\xi_i^f w_i^q(t=1)\}_{i=1}^{\infty}$ that [cf. (2.14)]

$$p_w^1(x) = \sum_{\sigma_{1,2}=\pm 1} P_1(\sigma_1, \sigma_2) \frac{1}{[2\pi\alpha D(t=1)]^{1/2}} \exp\left\{ -\frac{(x - \Delta(\sigma_1, \sigma_2))^2}{2\alpha D(t=1)} \right\} \tag{3.8}$$

Here $(f \neq q)$

$$P_1(\sigma_1, \sigma_2) = \Pr\{\sigma_1 = \xi_i^f \xi_i^q;\ \sigma_2 = \xi_i^f S_i(t=0)\} = \frac{1 + \sigma_1 \sigma_2 m^q(t=0)}{4} \tag{3.9}$$

Using (3.8) and the same arguments as above [see Eqs. (2.15)–(2.17)], we get from (3.6) that the residual overlap α-lim $r_N^f(t=2)$ again is the sum of two *correlated* Gaussian variables [cf. (2.17)]

$$r^f(t=2) = \mathcal{N}(0, 1) + 2p_w^1(x=0)\, r^f(t=1), \qquad f \neq q \tag{3.10}$$

By reasoning similar to the above [cf. (2.18)], but a bit lengthier calculation, one gets (see also ref. 5)

$$\begin{aligned} &\operatorname{Cov}[\mathcal{N}(0, 1) \cdot r^f(t=1)] \\ &\quad = m^q(t=2)\, m^q(t=1) \\ &\qquad + 2p_w^0(x=0) \left[m^q(t=0)\, m^q(t=2) + \frac{1 - (m^q(t=0))^2}{2} \right. \\ &\qquad \left. \times \sum_{\sigma=\pm 1} \sigma \operatorname{erf}\left(\frac{m^q(t=1) + 2\alpha\sigma p_w^0(x=0)}{[\alpha D(t=1)]^{1/2}} \right) \right] \end{aligned} \tag{3.11}$$

Equations (3.10) and (3.11) give the explicit formula for the variance of the random Gaussian residual overlaps at $t=2$ [cf. (2.19)]:

$$D(t=2) = 1 + (2p_w^1(x=0))^2\, D(t=1) + 4p_w^1(x=0) \operatorname{Cov}[\mathcal{N}(0, 1) \cdot r^f(t=1)] \tag{3.12}$$

4. THE THIRD STEP

We again have to repeat the analysis of the random variable α-lim $v_{N,i}^q(t)$ structure [see (2.5) and Eqs. (2.3), (3.1)], but now for $t=2$.

The same procedure as above [cf. (3.2), (3.3)] gives a new deviation from Gaussian random variable $[\alpha D(t=2)]^{1/2}\mathcal{N}(0,1)$ due to correlations:

$$v_i^q(t=2) = [\alpha D(t=2)]^{1/2}\mathcal{N}(0,1) + 2\alpha p_w^1(x=0) \cdot [S_i(t=1) + 2p_w^0(x=0)\, S_i(t=0)] \tag{4.1}$$

Compare Eqs. (2.8), (3.4), and (4.1). Then the probability density $h(x)$ for the noisy term α-lim $\eta_{N,i}^q(t=2)$ [see (3.1) and (3.4)] takes the form

$$h(x) = \sum_{\sigma_{1,2}=\pm 1} c(\sigma_1, \sigma_2)\frac{1}{[2\pi\alpha D(t=2)]^{1/2}}\exp\left[-\frac{(x-g(\sigma_1,\sigma_2))^2}{2\alpha D(t=2)}\right] \tag{4.2}$$

where

$$\begin{aligned} c(\sigma_1,\sigma_2) &= \Pr(\xi_i^q S_i(t=1)=\sigma_1;\ \xi_i^q S_i(t=0)=\sigma_2) \\ &= \frac{1+\sigma_1 m^q(t=1)}{2}\cdot\frac{1+\sigma_2 m^q(t=0)}{2} \\ g(\sigma_1,\sigma_2) &= 2\alpha p_w^1(x=0)[\sigma_1 + \sigma_2\cdot 2p_w^0(x=0)] \end{aligned} \tag{4.3}$$

Using the representation (3.1) for $t=3$ and Eqs. (4.1)–(4.3), we get [cf. (3.5)]

$$m^q(t=3) = \sum_{\sigma_{1,2}=\pm 1} c(\sigma_1,\sigma_2)\,\mathrm{erf}\left[\frac{m^q(t=2)+g(\sigma_1,\sigma_2)}{[\alpha D(t=2)]^{1/2}}\right] \tag{4.4}$$

Where $D(t=2)$ is defined by (3.12).

In ref. 5 we formulate a conjecture about a possible structure of the formula for the main overlap $m^q(t=n)$ for arbitrary n which gave us no hint about $t\to\infty$. That is why instead we consider the status of the AGS formula[9] for the main overlap m^q.

This formula has been derived by the methods of equilibrium statistical mechanics. Therefore, it is reasonable to consider that $m^q = m^q(t=\infty)$.

5. AMIT–GUTFREUND–SOMPOLINSKY FORMULA

Let $m^q = m$. Then we call the system of coupled equations

$$\begin{aligned} m &= \mathrm{erf}\left(\frac{m}{(\alpha R)^{1/2}}\right) \\ R^{1/2} &= 1 + \left(\frac{2}{\pi\alpha}\right)^{1/2}\exp\left(-\frac{m^2}{2\alpha R}\right) \end{aligned} \tag{5.1}$$

the AGS formula.[9] As mentioned above, this formula for the main overlap describes the final stage of evolution ($t = \infty$). Since for parallel dynamics (which we consider throughout this paper) the evolution for $t = \infty$ does not stop at the fixed configuration,[13] we can try to apply our observations about the dynamics of the main and the residual overlaps for this stage.

It is clear that under the parallel dynamics (1.4) at $t = \infty$ the system performs a random walk on some set $A(\xi^q)$ in the vicinity of the pattern ξ^q. This evolution is created by the internal noise due to the nonzero residual overlaps. It is very reasonable to suppose that at this stage we have:

1. $m^q_\infty(t+1) = m^q_\infty(t)$—stationarity of the main overlap.
2. $r^p_\infty(t+1) = r^p_\infty(t)$—strong stationarity of the internal noise ($p \neq q$).

Moreover, as follows from Eqs. (2.4), (2.7), and (3.10), the residual overlaps for any t are Gaussian random variables with zero mean and with the variance independent of p ($\neq q$).

3. $r^p_\infty(t) = \mathcal{N}(0, D)$, $p = 1, 2, \ldots$, and they are i.i.d.r.v.

Let $\mathbf{S}(t) \in A(\xi^q)$. Then, to calculate the next step $\mathbf{S}(t+1)$, we can use condition 3 and our first-step formulas (2.9), (2.17):

$$m^q_\infty(t+1) = \operatorname{erf}\left(\frac{m^q_\infty(t)}{(\alpha D)^{1/2}}\right) \tag{5.2}$$

$$r^p_\infty(t+1) = \mathcal{N}(0, 1) + \left(\frac{2}{\pi\alpha D}\right)^{1/2} \exp\left\{-\frac{[m^q_\infty(t)]^2}{2\alpha D}\right\} \cdot r^p_\infty(t) \tag{5.3}$$

Therefore, from condition 1 and Eq. (5.2) one gets

$$m^q_\infty(t) = \operatorname{erf}\left(\frac{m^q_\infty(t)}{(\alpha D)^{1/2}}\right) \tag{5.4}$$

Using (2.18) and (3.11), it is hard to predict how the correlation between $\mathcal{N}(0, 1)$ and $r^f(t)$ increases. But from condition 2 and Eq. (5.3) one gets that $\mathcal{N}(0, 1)$ and $r^f_\infty(t)$ have to be linearly correlated. Hence $\mathcal{N}(0, 1) = r^f_\infty(t)/\sqrt{D}$. Then, calculating the variance D defined by condition and Eq. (5.3), we obtain that

$$\sqrt{D} = 1 + \left(\frac{2}{\pi\alpha}\right)^{1/2} \exp\left\{\frac{[m^q_\infty(t)]^2}{2\alpha D}\right\} \tag{5.5}$$

For $m^q_\infty(t) = m$ and $D = R$, Eqs. (5.4) and (5.5) coincide with the system of AGS equations (5.1).

6. CONCLUDING REMARKS

For nonzero temperature $\theta \neq 0$ we have to use the stochastic equations (1.6). The heat-bath temperature noise (1.5) is completely uncorrelated with internal evolution and is unquenched. Therefore, to get the corresponding equations of dynamics for the main and residual overlaps, one has simply to average the equations for $\theta = 0$ over the linear noise (1.5), (1.6).[7]

For example, instead of Eqs. (2.9) and (2.17), one gets ($\beta = \theta^{-1}$)

$$m^q(t=1) = \frac{1}{(2\pi\alpha)^{1/2}} \int_{-\infty}^{+\infty} dx\, \mathrm{th}[\beta(x + m^q(t=0))] \exp\left(-\frac{x^2}{2\alpha}\right)$$

$$r^p(t=1) = \mathcal{N}(0,1) + r^p(t=0)\left(\frac{1}{2\pi\alpha}\right)^{1/2} \int_{-\infty}^{+\infty} dx \frac{\beta}{\mathrm{ch}^2\beta(x - m^q(t=0))} \times \exp\left(-\frac{x^2}{2\alpha}\right) \tag{6.1}$$

Similarly [cf. (3.5)] we get

$$m^q(t=2) = \sum_{\sigma=\pm1} \frac{1+\sigma m^q(t=0)}{2} \frac{1}{[2\pi\alpha D(t=1)]^{1/2}} \int_{-\infty}^{\infty} dx\, \mathrm{th}\, \beta\left[m^q(t=1) + x + \sigma\left(\frac{\alpha}{2\pi}\right)^{1/2} \int_{-\infty}^{+\infty} dy \frac{\beta}{\mathrm{ch}^2\beta(y - m^q(t=0))}\right] \exp\left(-\frac{x^2}{2\alpha D(t=1)}\right) \tag{6.2}$$

where [cf. (2.19)]

$$D(t=1) = 1 + \left\{\left(\frac{2}{\pi\alpha}\right)^{1/2} \int_{-\infty}^{+\infty} dx \frac{\beta \exp(-x^2/2\alpha)}{2\,\mathrm{ch}^2\beta(x - m^q(t=0))}\right\}^2 + 2m^q(t=1)\, m^q(t=0)\left(\frac{2}{\pi\alpha}\right)^{1/2} \int_{-\infty}^{+\infty} dx \frac{\beta \exp(-x^2/2\alpha)}{2\,\mathrm{ch}^2\beta(x - m^q(t=0))} \tag{6.3}$$

The same procedure applied to Eqs. (5.4) and (5.5) gives the well-known AGS equations for $\theta \neq 0$.[9]

Let us stress here that our derivation of the AGS formula is based on convincing but sloppy arguments which are far form rigorous. Moreover, they have to be modified for $\alpha > \alpha_c \approx 0.15$, where, according to (5.1), one gets $m^q = 0$, which is inconsistent with numerical results predicting $m^q(\alpha > \alpha_c) \neq 0$; see, e.g., refs. 4 and 9.

In conclusion, we remark that our results for $t=1, 2$ confirm the histograms obtained by computer simulations in ref. 4. For example, the histograms for $\{v_i^q(t=1)\}_i$ are symmetric despite becoming very broad for $\alpha > \alpha_c$ deviating from sound Gaussian form. The explanation comes from the explicit formula [cf. (3.4) and (2.14)]

$$v_i^q(t=1) = [\alpha D(t=1)]^{1/2}\, \mathcal{N}(0, 1) + \left(\frac{2\alpha}{\pi}\right)^{1/2} \exp\left\{-\frac{[m^q(t=0)]^2}{2\alpha}\right\} S_i(t=0) \tag{6.4}$$

Because of $\Pr\{S_i(t=0) = \pm 1\} = 1/2$ it is clear that $v_i^q(t=1)$ is a combination of two *symmetrically* shifted Gaussians and for small α this shift is very small due to the exponent in (6.4) (compare with ref. 4, Figs. 5a and 6a).

For

$$\xi_i^q v_i^q(t=1) = [\alpha D(t=1)]^{1/2} \xi_i^q \mathcal{N}(0, 1) + \left(\frac{2\alpha}{\pi}\right)^{1/2} \exp\left\{-\frac{[m^q(t=0)]^2}{2\alpha}\right\} \xi_i^q S_i(t=0) \tag{6.5}$$

the histogram is symmetric ($\Pr\{\xi_i^q = \pm 1\} = 1/2$) and close to Gaussian only for a small α (see ref. 4, Fig. 5b). For large α (in ref. 4, $\alpha = 0.16$), when the second term in the right-hand side of (6.5) becomes important, the random variable (6.5) is an *asymmetric* combination of the two Gaussians. These asymmetric shifts are due to initial conditions (2.2):

$$\Pr\{\xi_i^q S_i(t=0) = \pm 1\} = \begin{cases} [1 + m^q(t=0)]/2 \\ [1 - m^q(t=0)]/2 \end{cases} \tag{6.6}$$

This asymmetry is accumulating and increases with iterations because $\Pr\{\xi_i^q S_i(t) = +1\} > \Pr\{\xi_i^q S_i(t) = -1\}$; see (4.1) and (6.6). Finally, this produces a double-hill asymmetric histogram like that in ref. 4, Fig. 6b, for $t = 206$.

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